Creating Assessments that Promote Meaningful Learning within the Calculus Sequence

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What is meaningful learning?

Engaging students in meaningful learning (Novak, 2002):

- **Establish mathematics goals to focus learning**
- **Implement tasks that promote reasoning and problem solving**
- **Use and connect mathematical representations**
- **Pose powerful questions**
- **Elicit and use evidence of student thinking**

NCTM Process Standards (NCTM, 2014):

- **Problem Solving**
  - **Make sense of problems and persevere in solving them**
  - **Model with mathematics**
- **Reasoning & Proof**
  - **Reason abstractly and quantitatively**
  - **Construct viable arguments and critique the reasoning of others**
  - **Look for and express regularity in repeated reasoning**
- **Communications**
  - **Construct viable arguments and critique the reasoning of others**
- **Connections**
  - **Attend to precision**
  - **Look for and make use of structure**
- **Representations**
  - **Model with mathematics**

What do we want students to learn by doing homework?

- Applications to real world
- Perseverance (not giving up)
- Extend/explore beyond what is done in class on their own
- Recognize exceptions to rules
- Metacognition and understanding why they do what they do
- Socialization and collaboration
- Illuminate misconceptions
- Practicing skills - presenting skills and how to communicate mathematics (notation and organization)

Does an assessment item accomplish what we want it to?

We will look at a few items that were assigned as written homework. Consider the following questions:

- **What do you think is the learning goal for the item?**
- **What would students' response to the item look like?**
- **What would you be able to assess about students' understanding from these types of responses?**
- **Does the item provide a meaningful learning experience for students?**

From Calculus 2:

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at angle of 30 degrees along the edge of a diameter of the cylinder. Find the volume of the wedge.

Turning routine exercise into inquiry

Molly and Nikki are working on the following problem:

A wedge is cut out of circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge

(a) Molly wants to think of the wedge as a solid of cross sections to find the volume. Explain if Molly's method will work.

(b) Nikki wants to think of the wedge as a solid of revolution to find the volume. Explain if Nikki's method will work.

(c) Find the volume of the wedge.

Dorée (2017)

From Calculus 2:

Find $\displaystyle\int x\sqrt[3]{3-2x}\,dx.$

Using student counterexamples

In class, Michael and Kayla were working together on the following problem in class:

$$\text{Find } \int x\sqrt[3]{3-2x}\,dx.$$

(a) Kayla says, "u should be $(3-2x)$ because I always pick the most inside factor of a function as my u."

i. Will Kayla's substitution work in this case? Explain your reasoning.

ii. Does Kayla's idea work for all u-substitutions (if it does explain, if not give an example where it does not)?

(b) Michael says the u should be $\sqrt[3]{3-2x}$ because I always pick the most complicated factor of a function as my u."

i. Will Michael's substitution work in this case? Explain your reasoning.

ii. Does Michael's idea work for all u-substitutions (if it does explain, if not give an example where it does not)?

Vinsonhaler & Lynch (2020)

From Calculus 2:

A chain lying on the ground is 10m long and its mass is 80kg. How much work is required to raise the chain to a height of 6m?

Emphasize meaning

Jamie and Mike are working to understand the following problem:

A chain lying on the ground is 10 m long, and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

They went to the MARC for help, and found the solution could be found by evaluating the following integral:

$$W = 78.4 \int_0^6 (6 - x)\, dx$$

When they got home, they found they could not remember how they arrived at this solution. Help Jamie and Mike by explaining what each part of the integral represents in the context of the problem.

(a) Where did the 78.4 come from?

(b) What do the bounds represent?

(c) Where did the $(6 - x)$ come from?

(d) What does the dx represent?

(e) Why does the integral give Jamie and Mike the work required to raise one end of the chain to a height of 6 m?

From Calculus 3:

Find the equation of the tangent plane to $z = e^{x+y}$ *at the point* $(0,0,1)$.

Evaluating the reasoning of others' work

Charlie and Sammie were working to understand the following problem:

Find the equation of the tangent plane to $z = e^{x+y}$ *at the point* $(0, 0, 1)$.

They went to the MARC for help and worked with different tutors to come up with the following solutions:

Sammie:

$f(x,y) = e^{x+y} \qquad P = (x_0, y_0, z_0) = (0,0,1)$

$f_x = e^{x+y} \qquad f_x(0,0) = 1$

$f_y = e^{x+y} \qquad f_y(0,0) = 1$

$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$z - 1 = 1 \cdot (x - 0) + 1 \cdot (y - 0)$

$z = 1 + x + y$

Charlie:

$\bar{r}(u,v) = \langle u, v, e^{u+v} \rangle$

$P = (0,0,1) \Leftrightarrow (u,v) = (0,0)$

$\bar{r}_u = \langle 1, 0, e^{u+v} \rangle \qquad \bar{r}_u(0,0) = \langle 1, 0, 1 \rangle$

$\bar{r}_v = \langle 0, 1, e^{u+v} \rangle \qquad \bar{r}_v(0,0) = \langle 0, 1, 1 \rangle$

$\bar{n} = \bar{r}_u(0,0) \times \bar{r}_v(0,0) = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle -1, -1, 1 \rangle$

$\bar{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\langle -1, -1, 1 \rangle \cdot \langle x - 0, y - 0, z - 1 \rangle = 0$

$-x - y + z - 1 = 0$

Later in the semester, Charlie and Sammie decided to study for their Calculus III midterm together and showed each other their solutions to this problem. They found they could not remember the reasoning for their solutions and were confused by the different approaches. Help Charlie and Sammie by reviewing their solutions and answering their questions below:

(a) In Sammie's solution, where did $f(x, y)$ and P come from?

(b) In Sammie's solution, where did the formula $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ come from?

(c) In Charlie's solution, what is $\mathbf{r}(u, v)$?

(d) In Charlie's solution, what is $P = (0, 0, 1) \iff (u, v) = (0, 0)$ meant to convey?

(e) In Charlie's solution, what is $\mathbf{n}$ and how does $\mathbf{r}_u(0, 0) \times \mathbf{r}_v(0, 0)$ gives us $\mathbf{n}$?

(f) In Charlie's solution, where did the formula $\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ come from?

(g) Are Charlie and Sammie's answers equivalent? If so, why do both methods work and how are they related? If not, where did they go wrong and which method is correct?

(h) As Charlie looks over Sammie's solution he feels it looks very familiar and reminds him of Taylor polynomials in Calculus II. Sammie says it reminds her of the equation of a tangent line in Calculus I. Explain the connection between tangent planes, tangent lines, and Taylor polynomials.

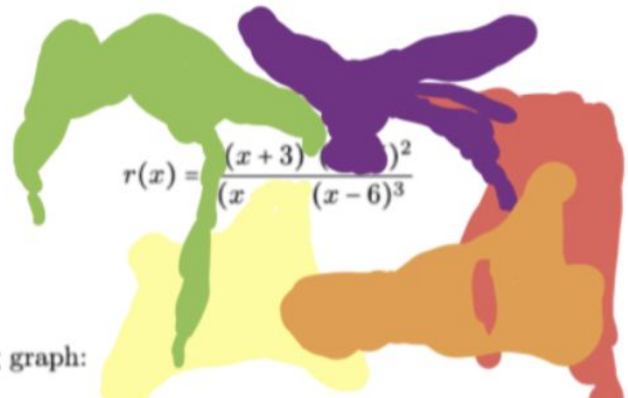
From Precalculus:

Graph the following function:

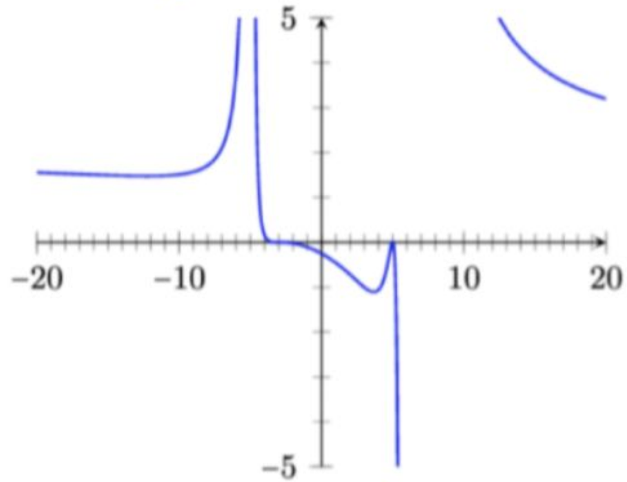
$$r(x) = \frac{2(x+3)^3(x-5)^2}{(x+5)^2(x-6)^3}$$

Smudged math

Lee came up with the following rational function while writing this homework up:



Which has the following graph:



As you may notice, Lee's two kids were home sick, and painting nearby. They spilled paint across the paper. Lee can recall there is a vertical asymptote at $x = -5$ and a horizaontal asymptote at $y = 2$, but nothing else about the rational function. Using what you know about rational functions and their graphs, find a plausible function for Lee's rational function graphed above. Explain the reason for your proposed rational function.

Liljedahl, 2020
Gordon, 2017

What will you think about the next time you write an assessment item?

- Ways to lead students down fruitful wrong avenues (make mistakes and learn from them)
- Experience more like what we do as mathematicians (try out different ideas)
- Demonstrate what they should be thinking while working on a problem
- Taking standard problems and making them memorable that sticks with the student.
- Give students solutions and make students make meaning of solutions

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Thanks!

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